On the detection of scalar hair

Narayan Banerjee*

Relativity and Cosmology Research Centre, Department of Physics, Jadavpur University, Calcutta-700032, India

Somasri Sen[†]

Mehta Research Institute

Naresh Dadhich[‡]

Inter University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune 411007, India

It is shown that the conclusion regarding the existence of a scalar hair for a black hole in a nonminimally coupled self interacting scalar tensor theory can be drawn from the that for a scalar field minimally coupled to gravity by means of a conformal transformation.

I. INTRODUCTION

The general belief that a black hole cannot have a scalar hair was disproved when it had been shown that in dilaton gravity, where a scalar field is nonminimally coupled to curvature, a black hole may indeed have a scalar hair [1]. This result motivated a lot of workers towards the search for the existence of a scalar hair in other nonminimally coupled scalar tensor theories of gravity. As there is no rigorous theorem to suggest in which cases a scalar hair might exist, the usual practice is to look at different examples. Unfortunately it is often quite difficult to find an exact solution for the spacetime metric with a nonminimally coupled scalar field. The exact solutions for a minimally coupled scalar field, on the other hand, are more readily found as the field equations are far less involved. Saa [2] proved a very useful theorem which states that the conclusions regarding the existence of a scalar hair for a wide class of nonminimally coupled scalar tensor theories can actually be drawn from the solutions for the metric for a spacetime with a scalar field minimally coupled to gravity. Saa's method proves to be extremely useful as it covers a very wide class of scalar tensor theories. This method had been generalized to include an elctromagnetic field [3] and also to find a counterexample of the no hair conjecture in an axially symmetric black hole where the scalar field is anisotropic and the spacetime is not asymptotically flat [4]. Although Saa's theorem is very general, it does not incorporate a scalar field with a self-interaction. Mayo and Bekenstein [5] included self interacting scalar fields, i.e. a scalar field with a potential, where the coupling between the curvature (R) and the scalar field (ϕ) has the form $\eta\phi^2 R$.

In this work we generalize Saa' work to include an electromagntic field or any $U(1)$ gauge field which has a trace free $(T = T^\mu_\mu = 0)$ energy momentum tensor and a self interaction term for the non minimally coupled scalar field. It is also generalization of the Mayo-Bekenstein case for we keep the coupling between scalar field and curvature free. We have been able to find a suitable redefinition of scalar field for the generalized framework as well and then the minimally and non minimally coupled scaler fields are described by the metrics that are conformally related.
In the next section we shall consider this transformation and demonstrate how it works which would be followed by a specific example in section III. We conclude with a discussion in section IV.

II. CONFORMAL TRANSFORMATION

We take a very general action of the form

$$S = \int \sqrt{-g}d^4x[f(\phi)R - h(\phi)\phi^{,\alpha}\phi_{,\alpha} + V(\phi) - F^{\alpha\beta}F_{\alpha\beta}], \tag{1}$$

where $f(\phi)$, $h(\phi)$, and $V(\phi)$ are functions of the scalar field (ϕ). $F_{\mu\nu}$ is the antisymmetric Maxwell tensor. The action is more general than that used by Saa [2] in the sense that it includes a potential $V(\phi)$ and an elctromagnetic field.

---

*Email address:narayan@juphys.ernet.in

[†]Email address: somasri@mri.ernet.in

[‡]Email address: nkd@iucaa.ernet.in

The scalar field couples with curvature as $f(\phi)R$, i.e., in a similar way as that used in Saa's ansatz and is more general than that used by Mayo and Bekenstein, where ϕ and R couple as $\eta\phi^2 R$, i.e., only with a quadratic ϕ.

The Einstein field equations with the action (1) are,

$$f(\phi)R_{\mu\nu} - h(\phi)\phi_{,\mu}\phi_{,\nu} - f_{,\mu;\nu} - \frac{1}{2}g_{\mu\nu}\Box f - \frac{1}{2}Vg_{\mu\nu} = T_{\mu\nu}, \tag{2}$$

where $T_{\mu\nu}$'s are the components of the energy momentum tensor for the electromagnetic field. We use the unit where $c = 1$ and $8\pi G = 1$, G being the Newtonian constant of gravitation. The wave equation for the scalar field is

$$2h\Box\phi + h'(\phi)\phi^{,\alpha}\phi_{,\alpha} + f'(\phi)R + V'(\phi) = 0. \tag{3}$$

where a prime indicates differentiation with respect to ϕ. This kind of non minimally coupled scalar field theories with a self interaction are very widely used at present in connection with the quintessence problem, i.e., to build up a cosmological model where the universe undergoes a late time acceleration [6]. From equation (2), the expression for the Ricci scalar R can be written as

$$R = \frac{h}{f}\phi^{,\alpha}\phi_{,\alpha} + 3\frac{\Box f}{f} - \frac{2V}{f}. \tag{4}$$

Now if we effect a conformal transformation of the form

$$\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{5}$$

where

$$\Omega^2 = f(\phi),$$

and redefine the scalar field as

$$\psi(\phi) = \sqrt{2}\int_{\phi_0}^{\phi} d\zeta\sqrt{\frac{3}{2}(\frac{d}{d\zeta}\ln f(\zeta))^2 + \frac{h(\zeta)}{f(\zeta)}}, \tag{6}$$

the action (1) takes the form

$$\bar{S} = \int d^4x\sqrt{-\bar{g}}[\bar{R} - \frac{1}{2}\psi^{,\alpha}\psi_{,\alpha} + \bar{V}(\psi) - \bar{F}^{\mu\nu}\bar{F}_{\mu\nu}]. \tag{7}$$

Here we have written

$$\bar{V} = \frac{V}{f^2}. \tag{8}$$

An overhead bar indicates quantities in the transformed version. The action (7) clearly represents the Einstein-Maxwell along with a minimally coupled scalar field ψ including a potential $\bar{V}$. The field equations become

$$\bar{R}_{\mu\nu} = \frac{1}{2}\psi^{,\alpha}\psi_{,\alpha} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{V} + \bar{T}_{\mu\nu}, \tag{9}$$

where $\bar{T}_{\mu\nu}$ represents the energy momentum tensor due to the transformed electromagnetic field. It should be noted that $\sqrt{-g}F^2 = \sqrt{-g}F^{\mu\nu}F_{\mu\nu}$ is invariant under this conformal transformation of the metric. The wave equation for the scalar field in this version looks like

$$\Box\psi + \frac{d\bar{V}}{d\psi} = 0, \tag{10}$$

The expression of Ricci scalar, followed from equation (9), in the transformed version is

$$\bar{R} = \frac{1}{2}\psi^{,\alpha}\psi_{,\alpha} - 2\bar{V}. \tag{11}$$

As the energy momentum tensor of the Maxwell field is trace free, its contribution does not appear in the expressions for the Ricci scalars (4) and (11) explicitly. The electromagnetic field enters into these scalars only through the

solutions for the metric tensor.

To determine the existence of a scalar hair (or any hair for that matter) for a black hole, one has to identify the site of the horizon and then see whether this horizon is regular with a non trivial scalar field (or the field corresponding to the particular kind of hair one is looking for). It is obvious that the field equations for a nonminimally coupled scalar field (2) are extremely involved, whereas the eqautions for a minimally coupled one (9) are much more amenable. Now if the solution for the metric $\bar{g}_{\mu\nu}$ with a minimally coupled scalar field with a potential $\bar{V}$ is known, the solution for the metric in any nonminimally coupled scalar tensor theory can be generated by the conformal transformation (5).

As these two metrics are conformally related, the site of the horizon, if there is any, will remain the same. For a given set of solutions for $\bar{g}_{\mu\nu}$ and ψ, the corresponding solutions and hence the nature of the surface designated to be the horizon for a nonminimally coupled theory with known $h(\phi)$ and $f(\phi)$ can be determined. One must however be careful that the potentials in these two cases are different, as given by the equation (8). It is naturally expected that with some regularity conditions on V, ϕ, f, h, and their powers, at the horizon, the singularity structure for the curvature scalar R and $\bar{R}$ would be the same. In other words, the presence or absence of a scalar hair for a nonminimally coupled scalar field with a potential V should be the same as that of a minimally coupled scalar field with a potential $\bar{V}$ given by the equation (8). Let us look at this with a Brans-Dicke type of coupling.

If we choose

$$f(\phi) = \phi; \qquad h(\phi) = \frac{\omega}{\phi}, \tag{12}$$

the action (1) represents the Brans-Dicke action [7] with an electromagnetic field and a self interaction term for the Brans–Dicke scalar field in the form $V(\phi)$. Without the Maxwell field, this is exactly the action used by many authors in order to obtain a reasonable accelerating model of the universe (see ref. [6]). With this choice, the expression for the Ricci scalar (4) reduces to

$$R = \omega\frac{\phi^{,\alpha}\phi_{,\alpha}}{\phi^2} + \frac{3\Box\phi}{\phi} - \frac{2V}{\phi}. \tag{13}$$

Now we replace $\Box\phi$ from equation (3) and use the transformations (5), (6) and (12) to get R in terms of the variables in the transformed version as

$$R = e^{\frac{\psi}{a}}\left(\frac{\omega}{a^2}\psi^{,\alpha}\psi_{,\alpha} - 2\bar{V} - \frac{3}{a}\frac{d\bar{V}}{d\psi}\right) \tag{14}$$

which in terms of $\bar{R}$ reads like

$$R = e^{\frac{\psi}{a}}\left[\frac{2\omega}{a^2}\bar{R} + 2\bar{V}\left(\frac{2\omega}{a^2} - 1\right) - \frac{3}{a}\frac{d\bar{V}}{d\psi}\right], \tag{15}$$

where $a = \sqrt{2\omega + 3}$. So it is expected that if ψ and the derivative of the potential $\bar{V}$ are regular at the horizon the regularity features of the curvature scalar are the same for R and $\bar{R}$.

III. A SPECIFIC EXAMPLE

Sudarsky [8] proved a very useful theorem showing that an asymptotically flat spherical black hole cannot support a Higgs type scalar hair. The relevant action is

$$S = \int d^4x\sqrt{-g}[\bar{R} - \frac{1}{2}\psi^{,\alpha}\psi_{,\alpha} - \bar{V}(\psi)], \tag{16}$$

in units where $8\pi G = 1$. This action can be readily identified with that given by (7) without the gauge field $F_{\mu\nu}$. With a very general spherically symmetric metric which is asymptotically flat, Sudarsky showed that there will be a regular horizon only if the scalar field is trivial, i.e., $\psi = constant$, and $\bar{V} = 0$. Now if we take a Brans–Dicke action with a potential $V(\phi)$, the corresponding metric tensor components can be derived with the help of the inverse transformations using equation (5). As these two spacetimes are conformally related, the site of the horizon would be

3

the same. Now Sudarsky's result shows that $\bar{R}$ will not be well behaved unless $\psi = constant$ and $\bar{V} = 0$,which clearly shows from equations (6), (8) that the scalar R as given by (15) will be well behaved only if $\phi = constant$ and $V = 0$. Thus a spherical black hole cannot support a Brans–Dicke scalar hair even if it is endowed with a self-interaction term.

Sudarsky's results are valid for an uncharged asymptotically flat spherical black hole. Hence in view of his theorem and the general results obtained in the previous section (equations (4 and (11)), it can be concluded that no uncharged asymptotically flat spherical black hole can have a scalar hair even if the scalar field is self interacting and nonminimally coupled to gravity.

IV. DISCUSSION

Dicke [9] used a conformal transformation of the metric tensor to rewrite the Brans–Dicke field equations in a form which is similar to that of a minimally coupled zero mass scalar meson field. Saa [2] gave a much more generalized version of the technique with the help of which the gravitational field equations for practically any nonminimally coupled scalar tensor theory can written as those for a massless minimally coupled scalar field. In the present work Saa's method is further generalized to include a self interaction term of the nonminimally coupled scalar field and any $U(1)$ gauge field. So from the solutions of a minimally coupled scalar field, the existence or the absence of a scalar hair can be predicted for a wide class of scalar tensor theories even including a potential. The non existence of a scalar hair for a self interacting scalar field was proved by Mayo and Bekenstein [5] for a particular type of coupling between the scalar field and the curvature. The method developed in the present work can be used for any kind of coupling. The method will fail in two cases. One is where $f(\phi)$ is negative, as the method depends crucially on the positivity of $f(\phi)$. But this perhaps should not be considered as a serious limitaion as in the weak field limit, f gives the inverse of the Newtonian gravitational constant, which is expected to be positive. It should be noted that although the method given in ref. [5] works where the coupling between R and scalar field is quadratic in ϕ, the corresponding $f(\phi)$ need not be restricted to positive values only. The second kind of coupling which escapes this method are the ones where the matter Lagrangian is also nonminimally coupled to the scalar field. The example being the dilaton gravity.
The inclusion of the electromagnetic field is also important, as in some cases where a scalar hair cannot grow independently but it may appear as a secondary one, i.e., growing on another hair like the electric charge. Once again the example is the dialton black hole [1].

V. ACKNOWLEDGEMENT

[1] D.Garfinkle, G.Horowitz and A.Strominger, *Phys. Rev.*, **D43**, 3140 (1991); G.W.Gibbons and and K.Maeda, *Nucl. Phys.* **B298**, 741 (1988).
[2] A. Saa, *J. Math. Phys.*, **37**, 2346 (1996).
[3] N. Banerjee and S. Sen, *Phys. Rev.*, **D58** 104024 (1998).
[4] S. Sen and N.Banerjee, *gr-qc/9809064* .
[5] A. Mayo and J. D. Bekenstein, *Phys.Rev.*, **D54**, 5509 (1996).
[6] O.Bertolami and P.J. Martins, *Phys. Rev.*, **D61**, 064007(2000); F.Perrotta, C.Baccigalupi and S. Matarrese, *Phys. Rev.* **D61**, 023507 (2000); N.Bartoto and M.Piertroni, *Phys. Rev.*, **D61**, 023518 (2000); S.Sen and T.Seshadri, *gr-qc/0007079* .
[7] C.H.Brans and R.H.Dicke, *Phys. Rev.* **121**, 925 (1961).
[8] D.Sudarsky, *Class. Quant. Grav.* **12**, 579 (1995).
[9] R.H.Dicke, *Phys. Rev.* **125**, 2163 (1962).